

Acoustic Scattering from a Sphere

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1 Introduction

This article derives the formula for scalar 2d scattering from a sphere. It uses my earlier article on scalar 2d scattering from a circle[1] as a starting point. That article in turn is based on a derivation on general 2d scattering[2]. Another derivation of many of these formulas can be found in an article by R. Kress available in print[3] and on the web[4]. Since this article is intended to be more of a summary than derivation, I refer the reader to the above articles and references therein for additional details and proofs.

The problem of scattering from a homogenous bounded object inside another homogenous region involves solving the Helmholtz Equation,

$$(\nabla^2 + k^2)u(\mathbf{x}) = 0, \tag{1}$$

where the pressure $p(\mathbf{x}, t)$ is related to u by the expression

$$p(x, t) = \Re \{ u(\mathbf{x})e^{i\omega t} \} . \tag{2}$$

The quantity k in Equation 1 is called the wave number and is related by the wavelength λ , angular frequency ω and speed of sound in the medium c by the relations

$$k = \frac{2\pi}{\lambda} \tag{3}$$

$$= \frac{\omega}{c} . \tag{4}$$

The wave number is actually a vector whose direction is the direction of propagation of the plane wave solutions to Equation 1. It is helpful to divide the total field u into a sum of an incident plane wave u_i and a scattered wave u_s with

$$u = u_i + u_s . \tag{5}$$

Since Equation 1 is linear and u_i satisfies this equation, u_s must satisfy it as well. Equation 1 has a unique solution in the regions outside the scatterer if u_s satisfies the Sommerfeld radiation condition at infinity

$$\lim_{r \rightarrow \infty} \left(\frac{\partial u_s}{\partial r} - ik u_s \right) = 0 \tag{6}$$

and appropriate boundary conditions on the surface of the scatterer.

1.1 Boundary Conditions

In this article, I will treat three boundary conditions. The first is for scattering from a “sound-soft” obstacle where the excess pressure is zero on the boundary (Dirichlet boundary condition). This isn’t the problem of immediate interest, but produces a good and simple test case. A similarly simple situation arises when the object is a hard scatterer so that the velocity is zero at the boundary. The final object I will treat is where the scatterer is a homogenous bounded scatterer. In that case, there are two boundary conditions which

relate the wave inside and outside the scatterer. Let u_D be u inside the scatterer. Continuity of the pressure and the normal velocity across the interface requires that

$$u = u_D \quad (7)$$

$$\frac{1}{\rho} \frac{\partial u}{\partial \hat{\mathbf{n}}} = \frac{1}{\rho_D} \frac{\partial u_D}{\partial \hat{\mathbf{n}}} \quad (8)$$

at the boundary. The quantities ρ and ρ_D are the respective densities of the material outside and inside the scatterer. The quantity $\frac{\partial u}{\partial \hat{\mathbf{n}}}$ is the normal derivative of u , where $\hat{\mathbf{n}}$ is the normal to the surface (pointing in to the scatterer),

$$\frac{\partial u}{\partial \hat{\mathbf{n}}} = \hat{\mathbf{n}} \cdot \nabla u. \quad (9)$$

In the case of a balloon, the problem which motivated this article, the boundary condition in Equation 8 seems reasonable, but Equation 7 is probably not going to hold. The membrane on the surface of the balloon provides a gradient in the pressure. To a first approximation, I'll assume the membrane doesn't have any surface waves (the fluids on either side can't exert transverse forces) and that the pressure boundary condition is just a constant pressure difference.

$$u = u_D - p_b, \quad (10)$$

where p_b is a constant on the surface of the scatterer.

1.2 Green's Function

The Green's function of Equation 1 is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}. \quad (11)$$

Application of Green's first and second identities to Equation 1 allow one to show that

$$u(\mathbf{x}) = \int_{\partial D} \left\{ u(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \hat{\mathbf{n}}(\mathbf{x}')} - \frac{\partial u}{\partial \hat{\mathbf{n}}}(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') \right\} ds(\mathbf{x}'), \quad (12)$$

where the integral is over the surface of D (which I'll represent as ∂D) and \mathbf{x} is in the region outside of the scatterer.

1.3 Far Field

The field u can be found far away from the scatterer by expanding the Green's function in Equation 11 in a Taylor series in $|\mathbf{x}|/|\mathbf{x}'|$. It is found that the wave is a spherical wave modulated by a factor $f(\hat{\mathbf{x}})$ which only depends on the direction of observation $\hat{\mathbf{x}}$.

$$u(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} f(\hat{\mathbf{x}}) \quad (13)$$

$$f(\hat{\mathbf{x}}) = \frac{1}{4\pi} \int_{\partial D} \left\{ u(\mathbf{x}') \frac{\partial e^{-ik\hat{\mathbf{x}} \cdot \mathbf{x}'}}{\partial \hat{\mathbf{n}}(\mathbf{x}')} - \frac{\partial u}{\partial \hat{\mathbf{n}}}(\mathbf{x}') e^{-ik\hat{\mathbf{x}} \cdot \mathbf{x}'} \right\} ds(\mathbf{x}') \quad (14)$$

Note that this expansion need not be applied to compute the field from the scattering problem. Equation 12 can be solved for the field anywhere in space outside of scatterer once u and its normal derivative on the surface are known.

1.4 Solutions in Spherical Coordinates

A separable solutions to Equation 1 can be found in spherical coordinates. There are two classes of solutions. The first set of solutions are valid everywhere in space, but don't satisfy the radiation condition. They are called the "complete solutions."

$$V_n^m(\mathbf{x}) = j_n(kx)Y_n^m(\hat{\mathbf{x}}), \quad (15)$$

where $x = |\mathbf{x}|$, j_n are the regular spherical Bessel functions, and Y_n^m are the spherical harmonics. The second class of solutions satisfy the Sommerfeld radiation condition, but are not regular at the origin. They are called the "radiating solutions."

$$W_n^m(\mathbf{x}) = h_n^{(1)}(kx)Y_n^m(\hat{\mathbf{x}}), \quad (16)$$

where h_n are the spherical Hankel functions with

$$h_n^{(1)}(z) = j_n(z) + iy_n(z). \quad (17)$$

The spherical Bessel functions are related to the regular Bessel functions by the relation

$$f_n(z) = \sqrt{\frac{\pi}{2z}} F_{n+1/2}(z), \quad (18)$$

where f_n is one of j_n , y_n , or h_n and F is one of J , Y , or H . The radiating solutions are complete in the sense that any radiating solution can be expanded as a linear combination of the terms in Equation 16.

$$u(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_n^m h_n^{(1)}(kx)Y_n^m(\hat{\mathbf{x}}). \quad (19)$$

The coefficients a_n^m are determined by the boundary conditions in the problem. Once these coefficients have been determined, the scattering amplitude in Equation 14 can be expressed as

$$f(\hat{\mathbf{x}}) = \frac{1}{k} \sum_{n=0}^{\infty} \frac{1}{i^{n+1}} \sum_{m=-n}^n a_n^m Y_n^m(\hat{\mathbf{x}}). \quad (20)$$

It will be useful later to have an expansion of the Green's function in Equation 11 in terms of these two solutions. It is

$$G(\mathbf{x}, \mathbf{x}') = ik \sum_{n=0}^{\infty} \sum_{m=-n}^n W_n^m(\mathbf{x}) \overline{V_n^m(\mathbf{x}')}, \quad (21)$$

where the overbar denotes the complex conjugate of an expression.

2 Potentials

It is convenient to solve scattering problems in terms of acoustic single-layer and acoustic double-layer potentials. Physically, these correspond to continuous layers of monopoles and dipoles on the surface of the scatterer. The potentials are solutions to Equation 1 inside and outside the scatterer and satisfy the Sommerfeldt radiation condition. The single-layer potential u and double-layer potential v away from the surface are

$$u(\mathbf{x}) = \int_{\partial D} \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') ds(\mathbf{x}') \quad (22)$$

$$v(\mathbf{x}) = \int_{\partial D} \rho(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \hat{\mathbf{n}}(\mathbf{x}')} ds(\mathbf{x}'). \quad (23)$$

Examination of Equation 12 shows that solutions of Equation 11 can be expressed as a combination of Equations 22 and 23 with the densities ρ being the values of u or its normal derivative on the boundary of the scatterer.

We are interested in the values of the potentials at the surface of the scatterer. Taking the limits approaching the surface requires some care because of a discontinuity in k . Taking this limit carefully, one finds that on the surface

$$u(\mathbf{x}) = \int_{\partial D} \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') ds(\mathbf{x}') \quad (24)$$

$$\frac{\partial u_{\pm}}{\partial \hat{\mathbf{n}}}(\mathbf{x}) = \int_{\partial D} \rho(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \hat{\mathbf{n}}(\mathbf{x}')} ds(\mathbf{x}') \mp \frac{1}{2} \rho(\mathbf{x}) \quad (25)$$

$$v_{\pm}(\mathbf{x}) = \int_{\partial D} \rho(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \hat{\mathbf{n}}(\mathbf{x}')} ds(\mathbf{x}') \pm \frac{1}{2} \rho(\mathbf{x}) \quad (26)$$

In addition, the normal derivative of v is continuous across the boundary ∂D . The + or - sign in the above formulas refers to the direction in which the limit is taken on the normal derivatives. A plus sign is the limit from the side in the direction of the normal to the surface. Formally,

$$\frac{\partial u_{\pm}}{\partial \hat{\mathbf{n}}} \equiv \lim_{h \rightarrow +0} \hat{\mathbf{n}}(\mathbf{x}) \cdot \nabla u(\mathbf{x} \pm h \hat{\mathbf{n}}(\mathbf{x})) . \quad (27)$$

With these limiting values of the potentials, it is simple to derive the following “jump relations” relating these potentials and their derivatives to each other on either side of the boundary.

$$u_+ = u_- \quad (28)$$

$$\frac{\partial u_+}{\partial \hat{\mathbf{n}}} - \frac{\partial u_-}{\partial \hat{\mathbf{n}}} = -\rho \quad (29)$$

$$v_+ - v_- = \rho \quad (30)$$

$$\frac{\partial v_+}{\partial \hat{\mathbf{n}}} = \frac{\partial v_-}{\partial \hat{\mathbf{n}}} \quad (31)$$

We will use these relations to enforce the boundary conditions for our various cases. The integrals appearing in the Equations 22, 23, 24, 25, and 26 can be written as the operators

$$(S\rho)(\mathbf{x}) = 2 \int_{\partial D} G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') ds(\mathbf{x}') \quad (32)$$

$$(K\rho)(\mathbf{x}) = 2 \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \hat{\mathbf{n}}(\mathbf{x}')} \rho(\mathbf{x}') ds(\mathbf{x}') \quad (33)$$

$$(K'\rho)(\mathbf{x}) = 2 \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \hat{\mathbf{n}}(\mathbf{x})} \rho(\mathbf{x}') ds(\mathbf{x}') \quad (34)$$

$$(T\rho)(\mathbf{x}) = 2 \frac{\partial}{\partial \hat{\mathbf{n}}(\mathbf{x})} \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial \hat{\mathbf{n}}(\mathbf{x}')} \rho(\mathbf{x}') ds(\mathbf{x}') \quad (35)$$

with these definitions, the “jump relations” can be written as

$$u_{\pm} = \frac{1}{2} S\rho \quad (36)$$

$$\frac{\partial u_{\pm}}{\partial \hat{\mathbf{n}}} = \frac{1}{2} K'\rho \mp \rho \quad (37)$$

$$v_{\pm} = \frac{1}{2} K\rho \pm \rho \quad (38)$$

$$\frac{\partial v_{\pm}}{\partial \hat{\mathbf{n}}} = \frac{1}{2} T\rho \quad (39)$$

Using this short-hand notation, the conditions on the solutions to Equation 1 for the region inside the scatterer can be written as

$$\begin{pmatrix} u \\ \partial u / \partial \hat{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} -K & S \\ -T & K' \end{pmatrix} \begin{pmatrix} u \\ \partial u / \partial \hat{\mathbf{n}} \end{pmatrix} . \quad (40)$$

For the exterior region this relation is

$$\begin{pmatrix} u \\ \partial u / \partial \hat{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} K & -S \\ T & ' \end{pmatrix} \begin{pmatrix} u \\ \partial u / \partial \hat{\mathbf{n}} \end{pmatrix} . \quad (41)$$

3 Scattering Theory

One of the first things a mathematician worries about in scattering problems is whether the problem as posed has a solution and whether that solution is unique. As physicists, the existence question may seem a bit quaint. Physically, it would seem impossible for there to be no solutions. Also, as we anticipated a single physical result, it would be unusual to have more than one possible solution.

However, there is a special case, that is physically meaningful and can cause numerical problems. Consider a system where the frequency of the sound matches an internal resonance in the scatterer. Since these disturbances go to zero on the boundary of the scatterer, they don't radiate and therefore don't contribute to the scattered wave. Any amount of these resonances added to the solution is therefore also a solution, and the equations will not have a unique answer.

3.1 Integral Equations

There are a number of different ways people use to solve the Equation 1 for scattering problems. One way to classify two general approaches are differential equation methods and integral equation methods. I will focus on integral equation methods in this article for a number of reasons:

1. By applying Green's theorem, integral equations can reduce the problem domain from \mathbb{R}^3 (3 dimensional space) to \mathbb{R}^2 (two dimensional space) on the boundary of the scatter. (This only applies to homogenous media.)
2. The exact Sommerfeld radiation condition is automatically taken into account. Differential equation methods need to do this by approximation radiation boundary conditions applied in the near field.
3. Finite difference approaches used in differential equation techniques involve small differences between large numbers. This can cause numerical difficulties unless the scattering surface is modeled accurately and great care is taken in doing the numerical derivatives.

The integral equations developed in Section 1 and the potentials defined in Section 2 provide a good starting point. Applying Equation 41 to u_s and Equation 40 to u_i ,

$$u_s = Ku_s - S \frac{\partial u_s}{\partial \hat{\mathbf{n}}} \quad (42)$$

$$u_i = -Ku_i + S \frac{\partial u_i}{\partial \hat{\mathbf{n}}} . \quad (43)$$

$$\frac{\partial u_s}{\partial \hat{\mathbf{n}}} = Tu_s - K' \frac{\partial u_s}{\partial \hat{\mathbf{n}}} \quad (44)$$

$$\frac{\partial u_i}{\partial \hat{\mathbf{n}}} = -Tu_i + K' \frac{\partial u_i}{\partial \hat{\mathbf{n}}} \quad (45)$$

Subtracting Equation 42 from Equation 43 and using Equation 5, one has

$$2u_i = Ku + S \frac{\partial u}{\partial \hat{\mathbf{n}}} + u . \quad (46)$$

Subtracting Equation 44 from Equation 45, one has

$$2 \frac{\partial u_i}{\partial \hat{\mathbf{n}}} = -Tu + K' \frac{\partial u}{\partial \hat{\mathbf{n}}} + \frac{\partial u}{\partial \hat{\mathbf{n}}} . \quad (47)$$

The existence of solutions to these equations depend on the smoothness of the surface, and in particular the functions u and $\frac{\partial u}{\partial \hat{\mathbf{n}}}$ on those surfaces. In the case of a sphere, we expect our surface to be smooth and not cause difficulties. We still face a problem of the uniqueness of the solutions near interior resonances (where Equation 46 has solutions for $u_i = 0$ or Equation 47 has solutions for $\frac{\partial u}{\partial \hat{\mathbf{n}}} = 0$). These solutions can be added to any other solution of the non-homogeneous equations to also give a valid solution. Luckily, these solutions don't radiate, and won't change our final computed scattered field. We will need to take some care with considering these limiting cases, however. One way the challenges of these internal resonances can be avoid is by solving a linear combination of Equation 46 and Equation 47.

$$2 \frac{\partial u_i}{\partial \hat{\mathbf{n}}} - 2iu_i = -Tu + K' \frac{\partial u}{\partial \hat{\mathbf{n}}} + \frac{\partial u}{\partial \hat{\mathbf{n}}} - iKu - iS \frac{\partial u}{\partial \hat{\mathbf{n}}} - iu . \quad (48)$$

3.2 Applying Boundary Conditions

There several classes of boundary conditions of interest in this case. I will first consider the simplest one, called the exterior Dirichlet problem. In this case, the function u has a known value f on the surface of the scattering. (For the simplest possible Dirichlet problem, $f = 0$.) This would be the acoustically soft sphere mentioned earlier. It is treated in Section 3.2.1.

Another interesting problem is where the normal displacement is zero at the boundary of the scatterer. This would be the case with a hard scatterer. It is treated in Section 3.2.2.

The final problem (and the one of current interest) would be the general case of a penetrable scatterer. This is the most complicated of the three cases because it involves considering both the interior and exterior solutions. It is treated in Section 3.2.3.

3.2.1 Dirichlet Problem

Integral Equation Approach For an acoustically soft sphere with $u = 0$ on the boundary, Equation 46 reduces to

$$2u_i = S \frac{\partial u}{\partial \hat{\mathbf{n}}} . \quad (49)$$

This problem could be solved in several straightforward ways. I will illustrate use a somewhat involved approach that has value of being generalizable for other boundary conditions and problems requiring numerical solutions.

The goal is to solve Equation 49 for $\partial u / \partial \hat{\mathbf{n}}$ on the surface of the sphere. For notational simplicity, let $\sigma = \partial u / \partial \hat{\mathbf{n}}$ on the surface of the sphere ∂D . Substituting this expression into Equation 49 with S as defined in Equation 32,

$$u_i = \int_{\partial D} G(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}') ds(\mathbf{x}') . \quad (50)$$

Since the Green's function has an expansion in terms of spherical harmonics, it makes sense to expand both G and σ in spherical harmonics. Let the sphere have radius a . From Equations 21, 15, and 16 one has

$$G(a\hat{\mathbf{x}}, a\hat{\mathbf{x}}') = ik \sum_{n=0}^{\infty} \sum_{m=-n}^n j_n(ka) h_n^{(1)}(ka) Y_n^m(\hat{\mathbf{x}}) \overline{Y_n^m(\hat{\mathbf{x}}')} . \quad (51)$$

I have also used the fact the j_n is real. We'll use the following expansion for σ .

$$\sigma(\hat{\mathbf{x}}') = \sum_{n=0}^{\infty} \sum_{m=-n}^n s_{nm} Y_n^m(\hat{\mathbf{x}}') \quad (52)$$

Since the Y_n^m functions are orthogonal to each other with the relation

$$\int Y_n^m(\hat{\mathbf{x}}) \overline{Y_{n'}^{m'}(\hat{\mathbf{x}})} d\hat{\mathbf{x}} = \delta_{m m'} \delta_{n n'} \quad (53)$$

the integral in Equation 50 can be done explicitly. The Kronaker δ 's appearing in Equation 53 reduce the quadruple sum to a double sum.

$$u_i = ik \sum_{n=0}^{\infty} \sum_{m=-n}^n s_{nm} j_n(ka) h_n^{(1)}(ka) Y_n^m(\hat{\mathbf{x}}) \quad (54)$$

The constants s_{nm} can be found by expanding the plane wave u_i in spherical coordinates as well. This expansion can be written as

$$u_i = e^{i\mathbf{k} \cdot \mathbf{x}} = 4\pi \sum_{n=0}^{\infty} i^n j_n(kx) \sum_{m=-n}^n Y_n^m(\hat{\mathbf{x}}) \overline{Y_n^m(\hat{\mathbf{k}})} . \quad (55)$$

Comparing the terms in Equation 54 with those in Equation 55, we see that

$$s_{nm} = \frac{4\pi}{k} i^{n-1} \frac{Y_n^{-m}(\hat{\mathbf{k}})}{h_n^{(1)}(ka)} . \quad (56)$$

Substituting the value for s_{nm} from Equation 56 into Equation 52, we have a solution for the normal derivative of u on the boundary.

$$\frac{\partial u}{\partial \hat{\mathbf{n}}}(a\hat{\mathbf{x}}') = \sigma(\hat{\mathbf{x}}') \quad (57)$$

$$= \frac{4\pi}{k} \sum_{n=0}^{\infty} \frac{i^{n-1}}{h_n^{(1)}(ka)} \sum_{m=-n}^n Y_n^m(\hat{\mathbf{x}}') \overline{Y_n^m(\hat{\mathbf{k}})} \quad (58)$$

$$= \frac{1}{ik} \sum_{n=0}^{\infty} i^n (2n+1) \frac{P_n(\hat{\mathbf{x}}' \cdot \hat{\mathbf{k}})}{h_n^{(1)}(ka)} \quad (59)$$

Equation 59 uses the identity

$$\frac{2n+1}{4\pi} P_n(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}) = \sum_{m=-n}^n Y_n^m(\hat{\mathbf{x}}) \overline{Y_n^m(\hat{\mathbf{y}})}. \quad (60)$$

Once σ is known, the far field can be computed from Equation 14. Using the expansion in Equation 55,

$$f(\hat{\mathbf{x}}) = -\frac{4\pi}{k} \int_{\partial D} \left[\sum_{n=0}^{\infty} \frac{i^{n-1}}{h_n^{(1)}(ka)} \sum_{m=-n}^n Y_n^m(\hat{\mathbf{x}}') \overline{Y_n^m(\hat{\mathbf{k}})} \right] \left[\sum_{n'=0}^{\infty} (-n' j_{n'}(ka)) \sum_{m'=-n'}^{n'} Y_{n'}^{m'}(\hat{\mathbf{x}}) \overline{Y_{n'}^{m'}(\hat{\mathbf{x}}')} \right] d\hat{\mathbf{x}}' \quad (61)$$

$$= \frac{4\pi i}{k} \sum_{m,n} \frac{j_n(ka)}{h_n^{(1)}(ka)} Y_n^m(\hat{\mathbf{x}}) \overline{Y_n^m(\hat{\mathbf{k}})} \quad (62)$$

$$= \frac{i}{k} \sum_{n=0}^{\infty} (2n+1) \frac{j_n(ka)}{h_n^{(1)}(ka)} P_n(\hat{\mathbf{x}} \cdot \hat{\mathbf{k}}). \quad (63)$$

A more general solution good in the near and far field can be computed from Equation 12 with the expansion in Equation 21 for the Green's function.

$$u(\mathbf{x}) = -4\pi i \sum_{n,n'} \frac{i^{n-1}}{h_n^{(1)}(ka)} j_{n'}(ka) h_{n'}(kx) \sum_{m=-n, m'=-n'}^{n, n'} \overline{Y_n^m(\hat{\mathbf{k}})} Y_{n'}^{m'} \int_{\partial D} Y_n^m(\hat{\mathbf{x}}') \overline{Y_{n'}^{m'}(\hat{\mathbf{x}}')} d\hat{\mathbf{x}}' \quad (64)$$

$$= -4\pi \sum_{n=0}^{\infty} i^n h_n^{(1)}(kx) \frac{j_n(ka)}{h_n^{(1)}(ka)} \sum_{m=-n}^n \overline{Y_n^m(\hat{\mathbf{k}})} Y_n^m(\hat{\mathbf{x}}) \quad (65)$$

$$= -\sum_{n=0}^{\infty} (2n+1) i^n h_n^{(1)}(kx) \frac{j_n(ka)}{h_n^{(1)}(ka)} P_n(\hat{\mathbf{x}} \cdot \hat{\mathbf{k}}). \quad (66)$$

Series Approach A simpler, but less generalizable way to solve this problem is to apply the boundary condition to series expansions of the incident plane wave and scattered wave at the surface of the sphere. An the surface,

$$e^{i\mathbf{k} \cdot a\hat{\mathbf{x}}'} = 4\pi \sum_{n,m} i^n j_n(ka) Y_n^m(\hat{\mathbf{x}}') \overline{Y_n^m(\hat{\mathbf{k}})} \quad (67)$$

$$u_s(a\hat{\mathbf{x}}') = \sum_{n,m} a_{nm} h_n^{(1)}(ka) Y_n^m(\hat{\mathbf{x}}') \quad (68)$$

For the soft sphere problem, the sum of these two series should be zero. Since the Y_n^m functions are linearly independent, the terms in each series must be equal to minus each other.

$$a_{nm} = -4\pi i^n \frac{j_n(ka)}{h_n^{(1)}(ka)} Y_n^m(\hat{\mathbf{k}}) \quad (69)$$

$$u_s(\hat{\mathbf{x}}) = -4\pi \sum_{n,m} i^n \frac{j_n(ka)}{h_n^{(1)}(ka)} h_n^{(1)}(kx) \overline{Y_n^m(\hat{\mathbf{k}})} Y_n^m(\hat{\mathbf{x}}) \quad (70)$$

$$= -\sum_{n=0}^{\infty} i^n (2n+1) \frac{j_n(ka)}{h_n^{(1)}(ka)} h_n^{(1)}(kx) P_n(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}) \quad (71)$$

3.2.2 Neuman Problem

Integral Equation Approach For the acoustically hard sphere with $\partial u / \partial \hat{\mathbf{n}} = 0$, Equation 47 reduces to

$$2 \frac{\partial u_i}{\partial \hat{\mathbf{n}}} = -T u. \quad (72)$$

It can be solved using a similar approach to the one given in Section 3.2.1. Substituting the expression for the operator T from Equation 35 into Equation 72,

$$\frac{\partial u_i}{\partial r} = \int_{\partial D} \frac{\partial^2 G(\mathbf{x}, \mathbf{x}')}{\partial r \partial r'} u(\mathbf{x}') ds(\mathbf{x}'). \quad (73)$$

Expanding the Green's function in spherical harmonics and taking the derivatives explicitly,

$$\frac{\partial^2 G(a\hat{\mathbf{x}}, a\hat{\mathbf{x}}')}{\partial r \partial r'} = ik^3 \sum_{n=0}^{\infty} \sum_{m=-n}^n j'_n(ka) h_n^{(1)'}(ka) Y_n^m(\hat{\mathbf{x}}) \overline{Y_n^m(\hat{\mathbf{x}}')}. \quad (74)$$

Expanding u in spherical harmonics,

$$u(\hat{\mathbf{x}}') = \sum_{n=0}^{\infty} \sum_{m=-n}^n t_{nm} Y_n^m(\hat{\mathbf{x}}'). \quad (75)$$

Using the same orthogonality relations as before,

$$\frac{\partial u_i}{\partial r} = ik^3 \sum_{n,m} t_{nm} j'_n(ka) h_n^{(1)'}(ka) Y_n^m(\hat{\mathbf{x}}'). \quad (76)$$

Expanding the radial derivative of the plane wave as in Equation 55,

$$\frac{\partial u_i}{\partial r} = 4\pi k \sum_{n=0}^{\infty} i^n j'_n(ka) \sum_{m=-n}^n Y_n^m(\hat{\mathbf{x}}') \overline{Y_n^m(\hat{\mathbf{k}})}. \quad (77)$$

Comparing the series in Equation 76 with the expansion in Equation 77, one can solve for t_{nm} .

$$t_{nm} = \frac{4\pi}{k^2} i^{n-1} \frac{Y_n^{-m}(\hat{\mathbf{k}})}{h_n^{(1)'}(ka)} \quad (78)$$

Substituting Equation 78 back into Equation 75 yields an expression for u on the surface.

$$u(\hat{\mathbf{x}}') = \frac{4\pi}{k^2} \sum_{n=0}^{\infty} \frac{i^{n-1}}{h_n^{(1)'}(ka)} \sum_{m=-n}^n \overline{Y_n^m(\hat{\mathbf{k}})} Y_n^m(\hat{\mathbf{x}}') \quad (79)$$

Plugging Equation 79 into Equation 14 with the plane wave expanded as before,

$$f(\hat{\mathbf{x}}) = \frac{4\pi i}{k} \sum_{n,n'=0}^{\infty} i^n (-i)^{n'} \frac{j'_n(ka)}{h'_n(ka)} \sum_{m,m'=-n,-n'}^{n,n'} \overline{Y_n^m(\hat{\mathbf{k}})} Y_{n'}^{m'}(\hat{\mathbf{x}}) \int_{\partial D} Y_n^m(\hat{\mathbf{x}}') \overline{Y_{n'}^{m'}(\hat{\mathbf{x}}')} ds(\hat{\mathbf{x}}') \quad (80)$$

$$= \frac{4\pi i}{k} \sum_{n=0}^{\infty} \frac{j'_n(ka)}{h'_n(ka)} \sum_{m=-n}^n \overline{Y_n^m(\hat{\mathbf{k}})} Y_n^m(\hat{\mathbf{x}}) \quad (81)$$

$$= \frac{i}{k} \sum_{n=0}^{\infty} (2n+1) \frac{j'_n(ka)}{h'_n(ka)} P_n(\hat{\mathbf{x}} \cdot \hat{\mathbf{k}}). \quad (82)$$

The general solution which is also valid in the near field comes from substituting Equation 79 into Equation 12 with the expansion in Equation 21 for the Green's function.

$$u(x) = -4\pi \sum_{n,m} i^n h_n^{(1)}(kx) \frac{j'_n(ka)}{h_n^{(1)'}(ka)} \overline{Y_n^m(\hat{\mathbf{k}})} Y_n^m(\hat{\mathbf{x}}) \quad (83)$$

$$= - \sum_{n=0}^{\infty} (2n+1) i^n h_n^{(1)}(kx) \frac{j'_n(ka)}{h_n^{(1)'}(ka)} P_n(\hat{\mathbf{x}} \cdot \hat{\mathbf{k}}). \quad (84)$$

Series Approach As the the soft sphere, the hard sphere problem can be solved by requiring that the sum of the series for the normal derivative of the incident and scattered waves by zero.

$$\frac{\partial}{\partial \hat{\mathbf{n}}} u_i = -\frac{\partial}{\partial x'} e^{i\mathbf{k} \cdot a \hat{\mathbf{x}}'} = -4\pi k \sum_{n,m} i^n j'_n(ka) Y_n^m(\hat{\mathbf{x}}') \overline{Y_n^m(\hat{\mathbf{k}})} \quad (85)$$

$$-\frac{\partial}{\partial \hat{\mathbf{n}}} u_s = \frac{\partial}{\partial x'} \sum_{n,m} a_{nm} h_n^{(1)}(kx') Y_n^m(\hat{\mathbf{x}}') \quad (86)$$

$$= k \sum_{n,m} a_{nm} h_n^{(1)'}(ka) Y_n^m(\hat{\mathbf{x}}') \quad (87)$$

$$a_{nm} = -4\pi i^n \frac{j'_n(ka)}{h_n^{(1)'}(ka)} h_n^{(1)} Y_n^m(\hat{\mathbf{k}}) \quad (88)$$

$$u_s(\hat{\mathbf{x}}) = -4\pi \sum_{n,m} i^n h_n^{(1)}(kx) \frac{j'_n(ka)}{h_n^{(1)'}(ka)} \overline{Y_n^m(\hat{\mathbf{k}})} Y_n^m(\hat{\mathbf{x}}) \quad (89)$$

$$= -\sum_{n=0}^{\infty} (2n+1) i^n h_n^{(1)}(kx) \frac{j'_n(ka)}{h_n^{(1)'}(ka)} P_n(\hat{\mathbf{x}} \cdot \hat{\mathbf{k}}) \quad (90)$$

3.2.3 Penetrable Scatterer

Integral Equation Approach The penetrable sphere can be computed with the series approach just as was done with the soft and hard spheres. Because of the orthogonality of the spherical harmonics, it leads to the same solution as the direct series solution which is outlined below. Since the extra algebra does little to enlighten the reader about the characteristics of the solution, it will be omitted here.

Series Approach I will break the field u at the boundary into three components: u_i (the initial plane wave), u_s (the scattered wave), and u_d (the wave inside the scatterer). Expressing each of these as a series as before,

$$u_i = 4\pi \sum_{n,m} i^n j_n(ka) Y_n^m(\hat{\mathbf{x}}') \overline{Y_n^m(\hat{\mathbf{k}})} \quad (91)$$

$$u_s = \sum_{n,m} a_{nm} h_n^{(1)}(ka) Y_n^m(\hat{\mathbf{x}}') \quad (92)$$

$$u_d = \sum_{n,m} b_{nm} j_n(k_d a) Y_n^m(\hat{\mathbf{x}}'). \quad (93)$$

Neglecting the pressure difference enabled by the balloon (imagine scattering from bubbles in water, for instance),

$$u_i + u_s = u_d. \quad (94)$$

We can recover our acoustically soft sphere solution by setting $b_{nm} = 0$. Since each term in the sum multiplies by the same factor of $Y_n^m(\hat{\mathbf{x}}')$, each term must be equal to zero.

$$4\pi i^n j_n(ka) \overline{Y_n^m(\hat{\mathbf{k}})} + a_{nm} h_n^{(1)}(ka) - b_{nm} j_n(k_d a) = 0 \quad (95)$$

$$\frac{4\pi i^n j_n(ka) \overline{Y_n^m(\hat{\mathbf{k}})} + a_{nm} h_n^{(1)}(ka)}{j_n(k_d a)} = b_{nm} \quad (96)$$

With or without the balloon, the radial velocities must match at the boundary as well (Equation 8).

$$\frac{1}{\rho} \frac{\partial}{\partial x'} (u_i + u_s) = \frac{1}{\rho_d} \frac{\partial}{\partial x'} u_d \quad (97)$$

Setting the sum of the derivative terms equal to zero yields

$$\frac{4\pi k}{\rho} i^n j'_n(ka) \overline{Y_n^m(\hat{\mathbf{k}})} + \frac{k}{\rho} a_{nm} h_n^{(1)'}(ka) = \frac{k_d}{\rho_d} b_{nm} j'_n(k_d a) \quad (98)$$

$$\frac{\frac{4\pi k}{\rho} i^n j'_n(ka) \overline{Y_n^m(\hat{\mathbf{k}})} + \frac{k}{\rho} a_{nm} h_n^{(1)'}(ka)}{\frac{k_d}{\rho_d} j'_n(k_d a)} = b_{nm} . \quad (99)$$

Setting the expression for b_{nm} in Equation 96 equal to the expression for b_{nm} in Equation 99,

$$\begin{aligned} \frac{4\pi i^n j_n(ka) \overline{Y_n^m(\hat{\mathbf{k}})} + a_{nm} h_n^{(1)}(ka)}{j_n(k_d a)} &= \frac{4\pi k \rho_d i^n j'_n(ka) \overline{Y_n^m(\hat{\mathbf{k}})} + k \rho_d a_{nm} h_n^{(1)'}(ka)}{k_d \rho j'_n(k_d a)} \\ a_{nm} \left[\frac{h_n^{(1)}(ka)}{j_n(k_d a)} - \frac{k \rho_d h_n^{(1)'}(ka)}{k_d \rho j'_n(k_d a)} \right] &= 4\pi i^n \overline{Y_n^m(\hat{\mathbf{k}})} \left[\frac{k \rho_d j'_n(ka)}{k_d \rho j'_n(k_d a)} - \frac{j_n(ka)}{j_n(k_d a)} \right] \\ a_{nm} &= 4\pi i^n \overline{Y_n^m(\hat{\mathbf{k}})} \frac{j_n(k_d a) k \rho_d j'_n(ka) - k_d \rho j'_n(k_d a) j_n(ka)}{k_d \rho j'_n(k_d a) h_n^{(1)}(ka) - j_n(k_d a) k \rho_d h_n^{(1)'}(ka)} . \end{aligned} \quad (100)$$

Plugging Equation 100 into Equation 92 (with a going to x),

$$u_s(\mathbf{x}) = 4\pi \sum_{n,m} i^n \frac{j_n(k_d a) k \rho_d j'_n(ka) - k_d \rho j'_n(k_d a) j_n(ka)}{k_d \rho j'_n(k_d a) h_n^{(1)}(ka) - j_n(k_d a) k \rho_d h_n^{(1)'}(ka)} h_n^{(1)}(kx) \overline{Y_n^m(\hat{\mathbf{k}})} Y_n^m(\hat{\mathbf{x}}) \quad (101)$$

$$= \sum_n i^n (2n+1) \frac{k \rho_d j'_n(ka) j_n(k_d a) - k_d \rho j'_n(k_d a) j_n(ka)}{k_d \rho j'_n(k_d a) h_n^{(1)}(ka) - k \rho_d h_n^{(1)'}(ka) j_n(k_d a)} h_n^{(1)}(kx) P_n(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}) \quad (102)$$

$$f(\hat{\mathbf{x}}') = -\frac{i}{k} \sum_n (2n+1) \frac{k \rho_d j'_n(ka) j_n(k_d a) - k_d \rho j'_n(k_d a) j_n(ka)}{k_d \rho j'_n(k_d a) h_n^{(1)}(ka) - k \rho_d h_n^{(1)'}(ka) j_n(k_d a)} P_n(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}}) . \quad (103)$$

Note that if $k = k_d$ and $\rho = \rho_d$, the scattered wave has an amplitude of zero as would be expected. In the limit of k_d small, Equation 102 reduces to that for scattering from a hard sphere. In the limit of k_d large, Equation 102 reduces to that for scattering from a soft sphere.

4 Balloon Scattering

In this section, I'll generalize the results from Section 3 to that from an idealized balloon. To make generalizations of this calculations explicit, I'll start by outlining the assumptions of the theory in this section.

1. The balloon is a perfect sphere.
2. The balloon membrane is negligibly thin and has the sole effect of providing a pressure difference between the gas on the outside of the balloon and inside the balloon.
3. The gasses inside and outside the balloon have uniform density and temperature.

5 Results

In this section, I have included some numerical results for the formulas derived in this paper. They are divided into three groups, the acoustically soft sphere, the acoustically hard sphere, and penetrable scatterers.

5.1 Soft Sphere

The following Matlab functions were used to plot scattering from a soft sphere of radius a . The first thing I needed were some short functions to compute spherical Bessel functions. The function in the `sphj.m` computes the spherical Bessel function of the first kind

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) . \quad (104)$$

```

function j=sphj(n,x)
% compute the spherical bessel function of the first kind
% of order n at the point x
j=sqrt(pi/(2*x))*besselj(n+0.5,x);
end

```

Similarly, the file function in `sphh.m` computes the spherical Hankel function of the first kind

$$h_n(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}(x) . \quad (105)$$

```

function h=sphh(n,x)
% compute the spherical hankel function of the first kind
% of order n at the point x
h=sqrt(pi/(2*x))*besselh(n+0.5,x);
end

```

5.1.1 Far Field

The function `soft.m` computes the cross section $|f|^2$ for a soft sphere of radius a (measured in wavelengths).

```

function [theta,intensity]=soft(a)
% *** soft(a) ***
% compute scattering from a soft sphere of radius a
% at the angle theta using nmax as the maximum value
% of n in the sum.
% The terms in the series oscillate up to about n=ka and then
% decrease rapidly. Taking 2ka terms, should be more than sufficient
% for large spheres. It always uses a minimum of 5 terms.
theta=0:0.01:pi; % scattering angle in radians
k=2*pi; % wave number with wavelength of 1
nmax=2*k*a+5;
nary=0:nmax;
nterm=2*nary+1;
jn=sphj(nary,k*a);
hn=sphh(nary,k*a);
% legendre calculates the associated legendre polynomials.
% I'm only interested in the m=0 one
for term=nary
    lpn=legendre(term,cos(theta));
    pn(term+1,:)=lpn(1,:);
end
for term=1:size(theta,2)
    intensity(term)=abs(i/k*sum(nterm.*jn./hn.*pn(:,term)'))^2;
end
end

```

It is somewhat slow to run this program in Matlab for large spheres, but works fine for spheres up to 10 wavelengths or so. Figures 1 through 4 are the squares of the scattering amplitude for acoustically soft spheres of various wavelengths.

These plots generally look okay to me, but I'm worried about a factor of 2 in the normalization. For a large sphere, I would expect the total cross section

$$\sigma = \int |f(\Omega)^2 d\Omega \quad (106)$$

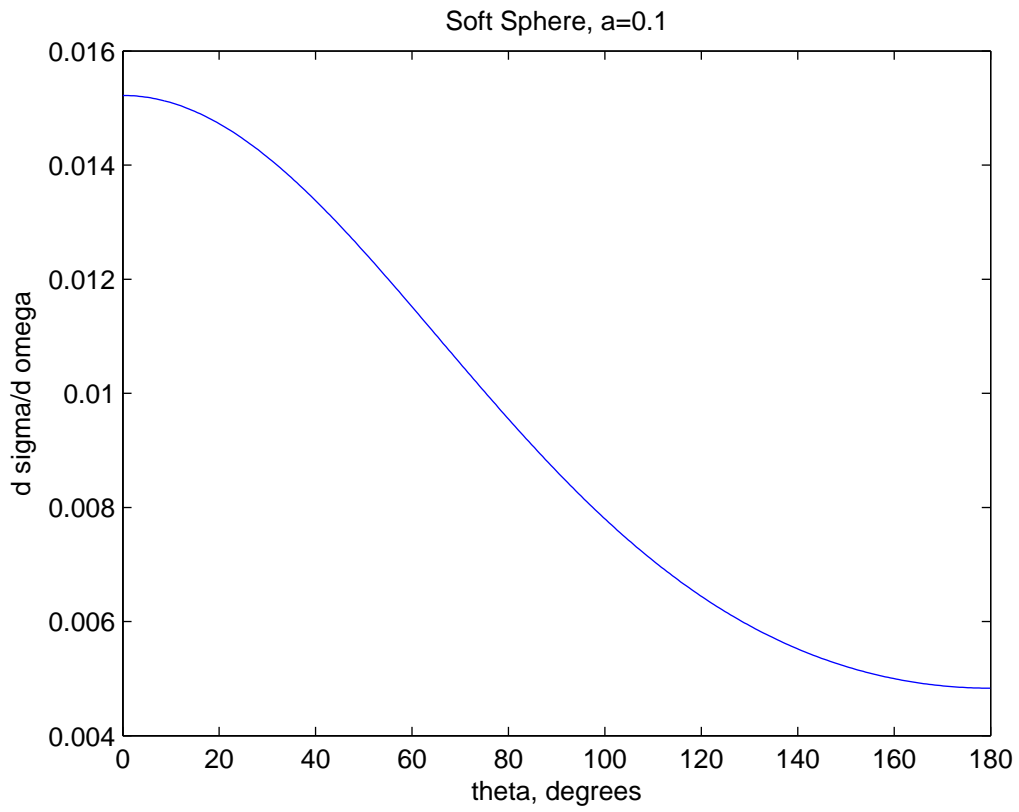


Figure 1: Differential cross section for scattering from an acoustically soft sphere of radius 0.1 wavelengths.

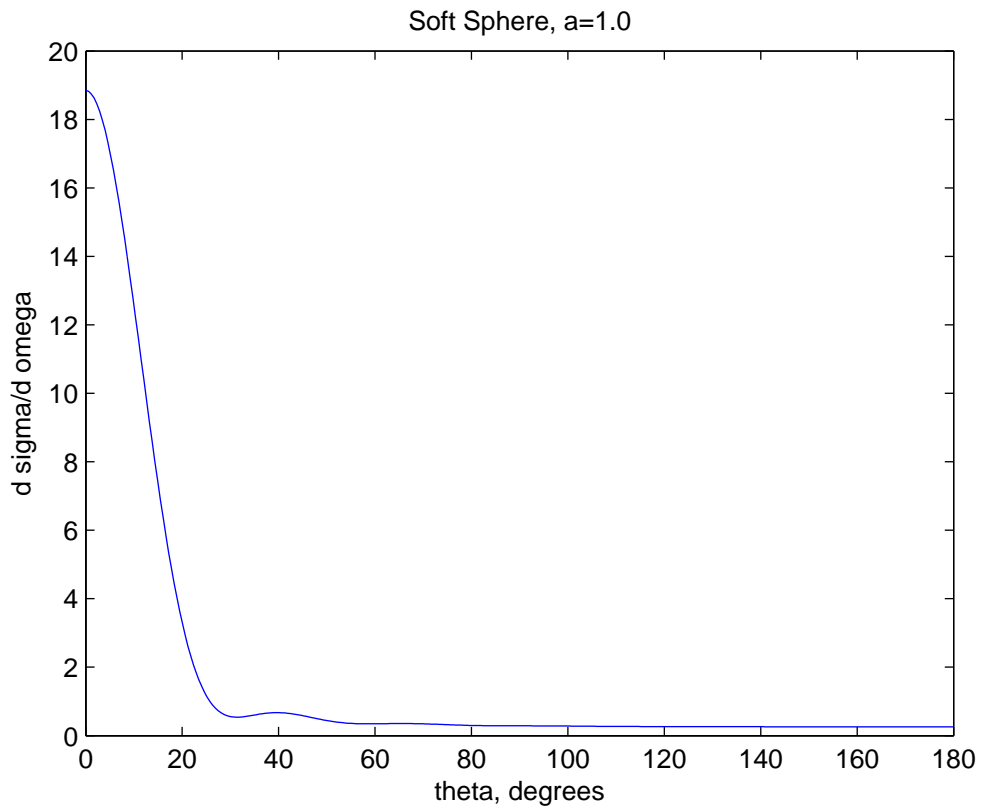


Figure 2: Differential cross section for scattering from an acoustically soft sphere of radius 1.0 wavelength.

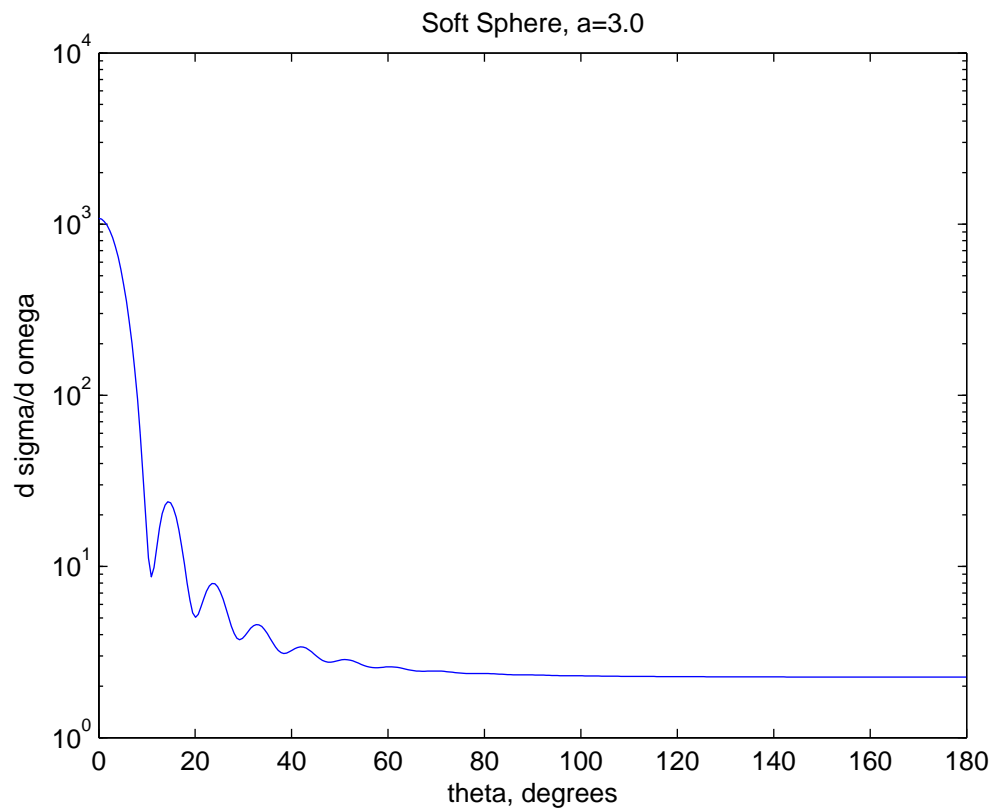


Figure 3: Differential cross section for scattering from an acoustically soft sphere of radius 3.0 wavelengths.

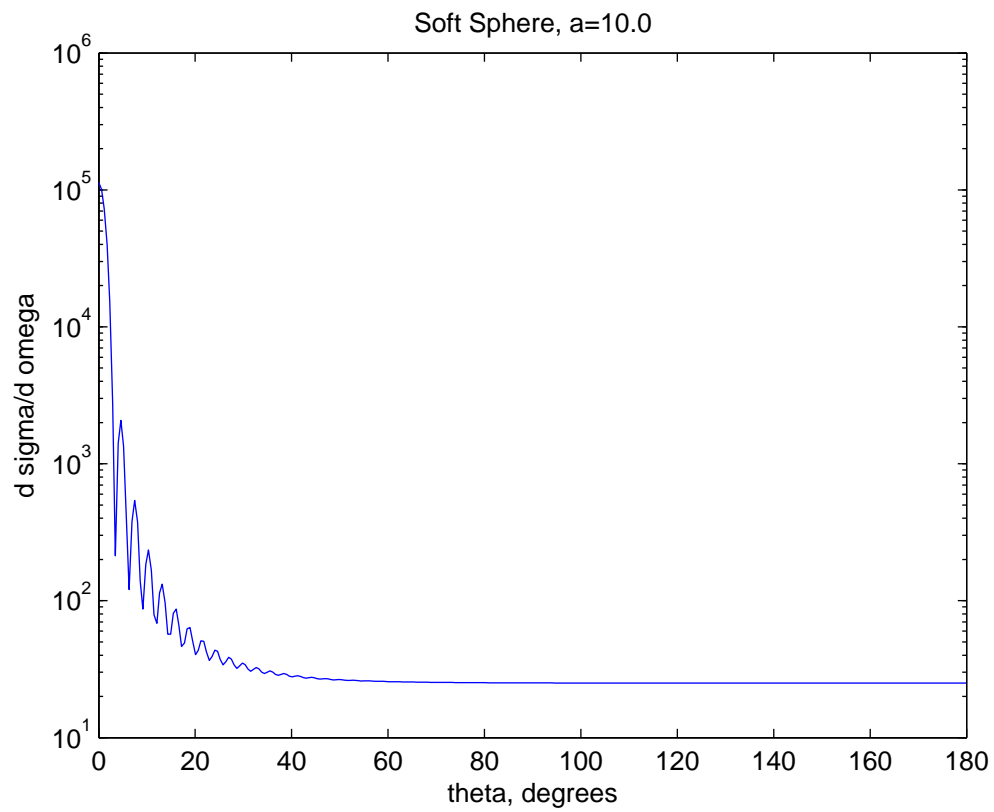


Figure 4: Differential cross section for scattering from an acoustically soft sphere of radius 10.0 wavelengths.

to be equal to the cross sectional area of the sphere

$$A = \pi a^2 . \tag{107}$$

If I integrate $|\cdot|^2/\pi$ numerically for spheres of radius 10 and 20, I get 10.28² and 20.02².

```
>> [theta10,dsigma10]=soft(10);
>> [theta20,dsigma20]=soft(20);
>> sqrt(sum(dsigma10.*sin(theta10))*(theta(2)-theta(1)))
ans =
    10.2574
>> sqrt(sum(dsigma20.*sin(theta20))*(theta(2)-theta(1)))
ans =
    20.0186
```

I can also do these integrals analytically by taking the large argument expansions for the Bessel and Hankel functions. Abramowitz and Stegun[5] formula 10.1.16 gives a series expansion for $h_n^{(1)}(z)$ as

$$h_n^{(1)}(z) = i^{-n-1} z^{-1} e^{iz} \sum_{k=0}^n (n+1/2, k) (-2iz)^{-k} . \tag{108}$$

For large x, the dominant term in this series will be the k=0 term. Therefore, the asymptotic values for j_n and $h_n^{(1)}$ for large x are

$$h_n(x) = \frac{e^{i(x-\pi/2-n\pi/2)}}{x} \tag{109}$$

$$j_n(x) = \frac{\cos\left(x - \frac{\pi}{2} - \frac{n\pi}{2}\right)}{x} . \tag{110}$$

Note that the asymptotic expansion for h_n in Equation 109 can be substituted into Equation 63 to give Equation 66 and into Equation 82 to give Equation 84, providing a good cross check on those sets of formulas.

The integrated cross section can be computed using the orthogonality relation for the Legendre polynomials

$$\int_{-1}^1 P_n(x)P_m(x) = \frac{2}{2n+1} \delta_{n,m} . \tag{111}$$

The integrated square of Equation 66 is then

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f|^2 d\Omega = \frac{2}{k^2} \sum_{n=0}^{\infty} (2n+1) \frac{(j_n(ka))^2}{\left|h_n^{(1)}(ka)\right|^2} . \tag{112}$$

I'd like to make a further reduction of this analytically, but haven't been able to figure it out yet.

5.1.2 Near Field

I also implemented a computation of the near field intensity in the file `softnear.m`. It takes a very long time to run in matlab and is much more practical in a compiled language.

```
function [x,y,intensity]=softnear(a)
% *** soft(a) ***
% compute scattering from a soft sphere of radius a
% at the angle theta using nmax as the maximum value
% of n in the sum.
% The terms in the series oscillate up to about n=ka and then
% decrease rapidly. Taking 2ka terms, should be more than sufficient.
```

```

% It always uses a minimum of 5 terms.
x=-10:.1:10;
y=x;
k=2*pi; % wave number with wavelength of 1
nmax=2*k*a+5;
nary=0:nmax;
nterm=2*nary+1;
jn=sphj(nary,k*a);
hn=sphh(nary,k*a);
% legendre calculates the associated legendre polynomials.
% I'm only interested in the m=0 one
for ix=1:size(x,2)
    for iy=1:size(y,2)
        r=sqrt(x(ix)^2+y(iy)^2);
        if(r<a)
            intensity(ix,iy)=0;
        else
            for term=nary
                lpn=legendre(term,x(ix)/r);
                pn(term+1,:)=lpn(1,:);
            end
            hnx=sphh(nary,k*r);
            intensity(iy,ix)=abs(exp(i*k*x(ix))-...
                sum(i.^nary.*nterm.*jn./hn.*hnx.*pn'))^2;
        end
    end
end
end
end

```

Figure 5 is the output of an equivalent FORTRAN program for the total near field intensity near a soft sphere of radius 2.5 wavelengths.

5.2 Hard Sphere

The formulas for the hard sphere involve derivatives of bessel functions. These were computed using the differentiation formula 10.1.22 in Abramowitz and Stegun[5],

$$\frac{d}{dz} f_n(z) = \frac{n}{z} f_n(z) - f_{n+1}(z), \quad (113)$$

where the function f can be j_n , y_n , or h_n . The function `sphjp` computes $d/dx f_n(x)$.

```

function jp=sphjp(n,x)
% compute the derivative of the spherical bessel function of the
% first kind of order n at the point x
jp=n/x.*sphj(n,x)-sphj(n+1,x);
end

```

The function `sphhp` compute $d/dx_n^{(1)}(x)$.

```

function hp=sphhp(n,x)
% compute the derivative of the spherical hankel function of the
% first kind of order n at the point x
hp=n/x*sphh(n,x)-sphh(n+1,x);
end

```

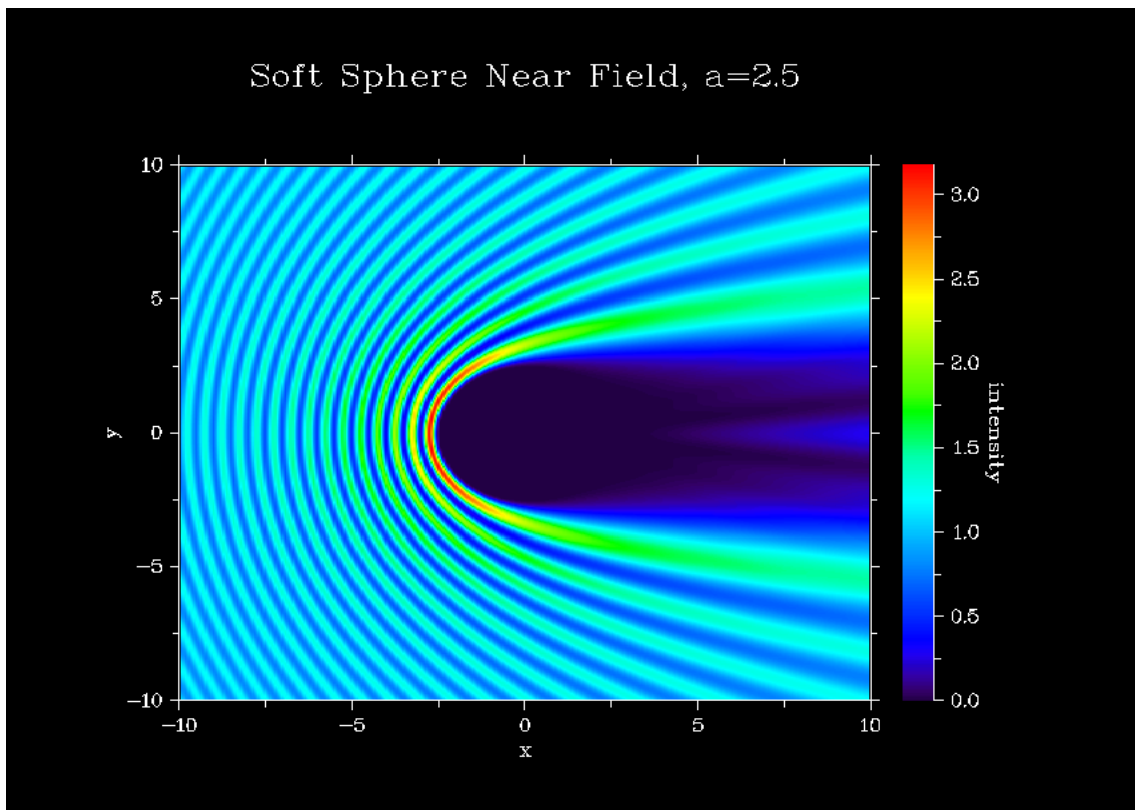


Figure 5: Near field intensity for scattering from a soft sphere of radius 2.5 wavelengths.

5.2.1 Far Field

The function `hard` computes the differential cross section for scattering from a hard sphere.

```
function [theta,intensity]=soft(a)
% *** soft(a) ***
% compute scattering from a soft sphere of radius a
% at the angle theta using nmax as the maximum value
% of n in the sum.
% The terms in the series oscillate up to about n=ka and then
% decrease rapidly. Taking 2ka terms, should be more than sufficient.
% It always uses a minimum of 5 terms.
theta=0:0.01:pi; % scattering angle in radians
k=2*pi; % wave number with wavelength of 1
nmax=2*k*a+5;
nary=0:nmax;
nterm=2*nary+1;
jnp=sphjp(nary,k*a);
hnp=sphhp(nary,k*a);
% legendre calculates the associated legendre polynomials.
% I'm only interested in the m=0 one
for term=nary
    lpn=legendre(term,cos(theta));
    pn(term+1,:)=lpn(1,:);
end
for term=1:size(theta,2)
    intensity(term)=abs(i/k*sum(nterm.*jnp./hnp.*pn(:,term)'))^2;
end
end
```

For large spheres, this gives an integrated result close to that of the small sphere. Figures 6 through 9 are the squares of the scattering amplitudes for acoustically hard spheres.

5.2.2 Near Field

The total intensity in the near field can be computed using the same FORTRAN program which did the near field for the soft sphere. The results for a 2.5 radius hard sphere are shown in Figure 10. The incident field is coming from the left.

5.3 Penetrable Scatters

5.3.1 Far Field

I implemented Equation 101 in the Matlab file `penetrate.m`. This hasn't been as extensively checked as the `soft.m` and `hard.m`. It may not agree with them well in the cases where k_d is very large and very small as it should. However, the FORTRAN version of this used for near field calculations agrees very well with the cases of k_d small, unity, and large.

```
function [theta,intensity]=penetrate(a,kf,rhof)
% *** penetrate(a) ***
% compute scattering from a penetrable sphere of radius a
% kf=k_d/k, here k is the wave number (2 pi/lambda) of the gas in the
% exterior volume and k_d is the wave number inside the sphere.
% rhof=rho_d/rho where rho is the density of the gas in the
% exterior volume and rho_d is the density of the gas inside the sphere.
% The terms in the series oscillate up to about n=ka and then
```

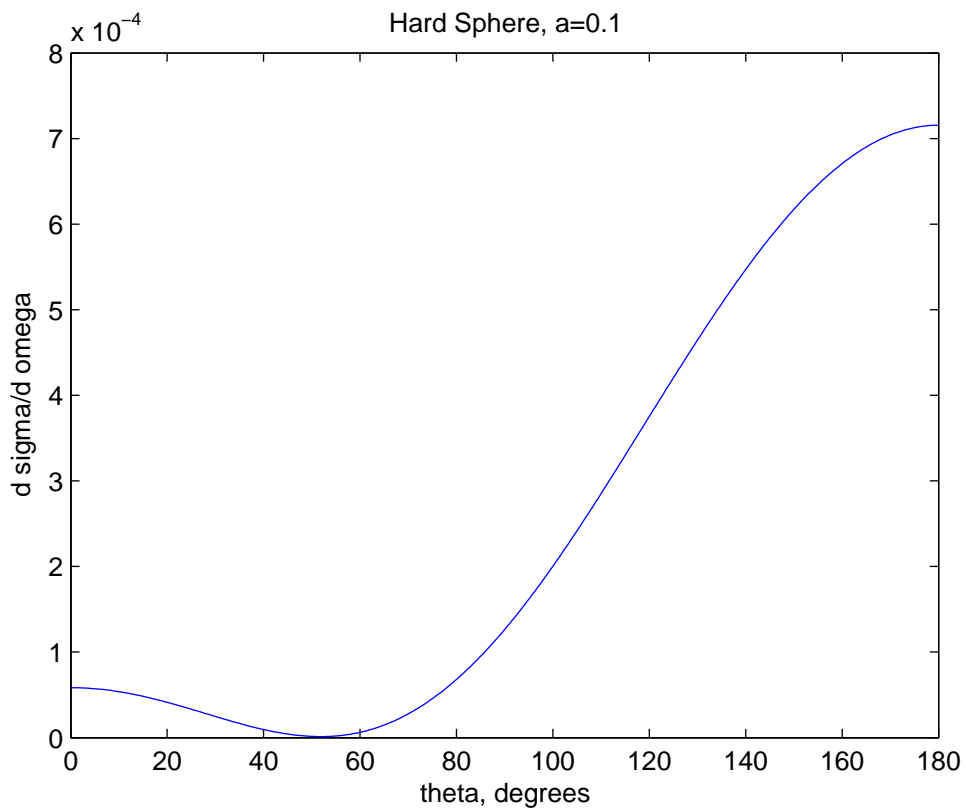


Figure 6: Differential cross section for scattering from an acoustically hard sphere of radius 0.1 wavelengths.

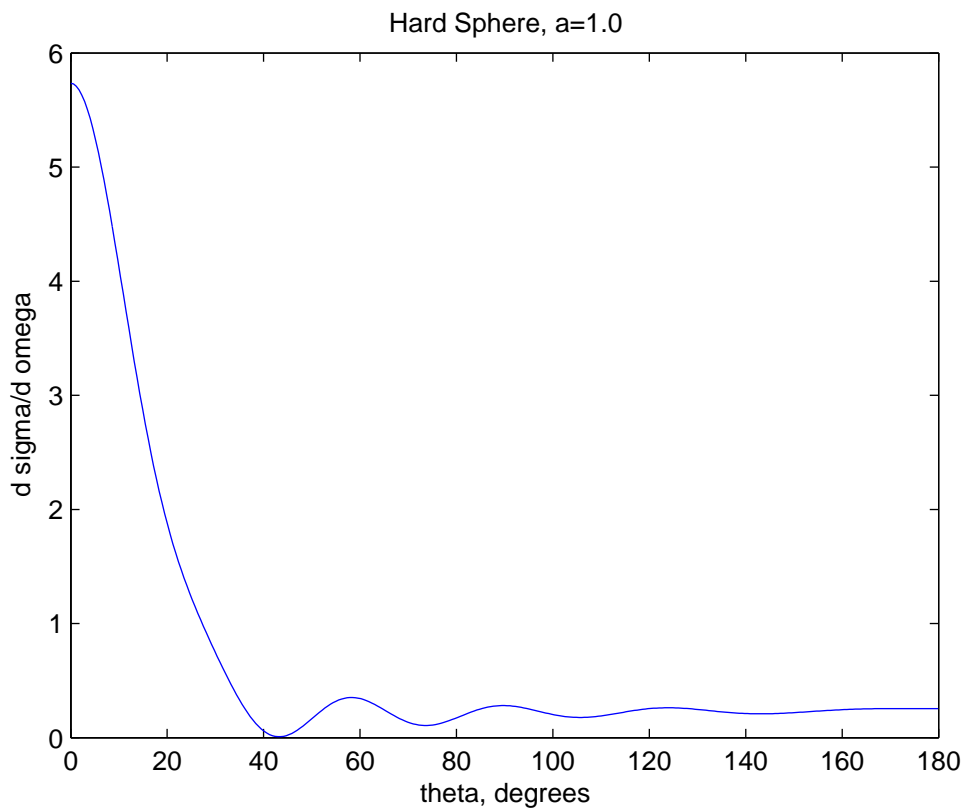


Figure 7: Differential cross section for scattering from an acoustically hard sphere of radius 1 wavelength.

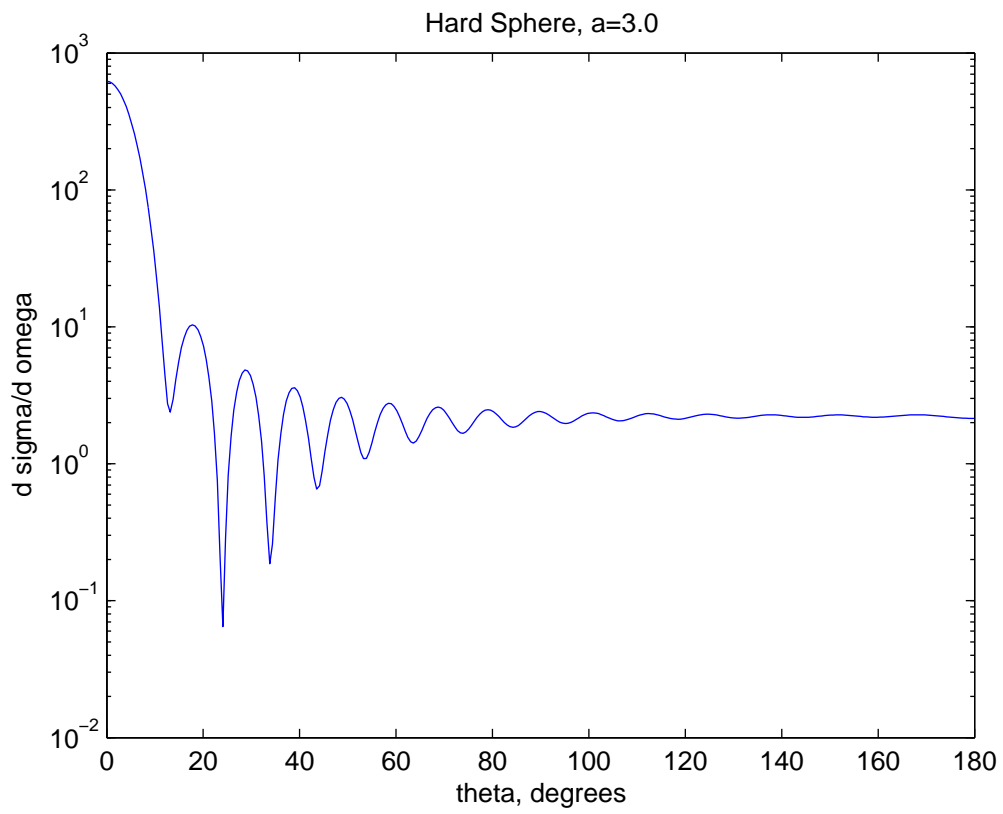


Figure 8: Differential cross section for scattering from an acoustically hard sphere of radius 3 wavelengths.

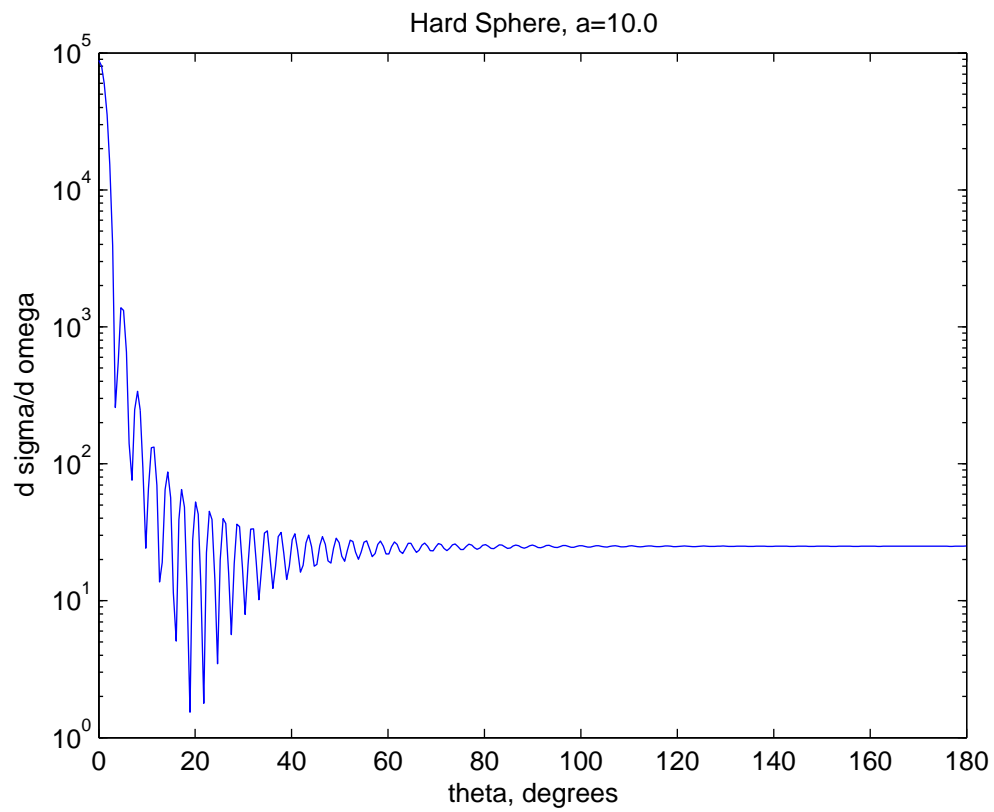


Figure 9: Differential cross section for scattering from an acoustically hard sphere of radius 10 wavelengths.

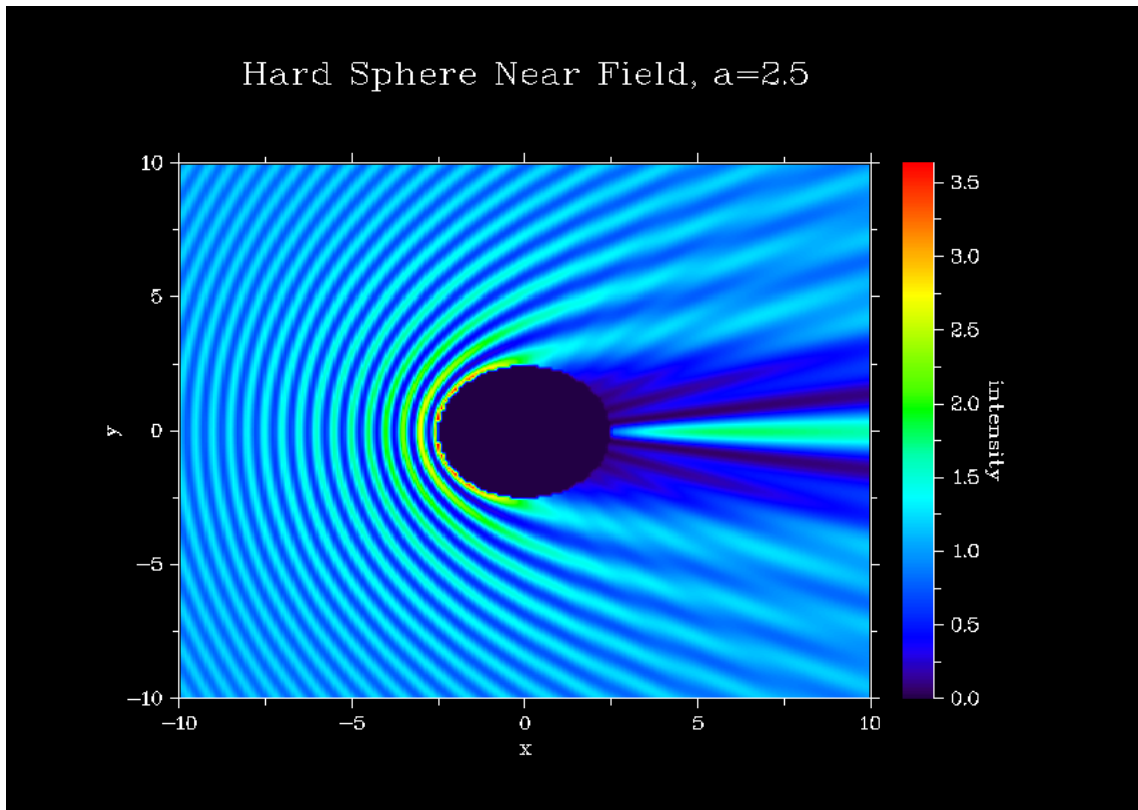


Figure 10: Total intensity near a 2.5 wavelength radius hard sphere.

```

% decrease rapidly. Taking 2ka terms, should be more than sufficient.
% It always uses a minimum of 5 terms.
theta=0:0.01:pi; % scattering angle in radians
k=2*pi; % wave number with wavelength of 1
ka=k*a;
kda=k*kf*a;
nmax=2*k*a+5;
nary=0:nmax;
nterm=2*nary+1;
jn=sphj(nary,ka);
hn=sphh(nary,ka);
jnp=sphjp(nary,ka);
hnp=sphhp(nary,ka);
jnd=sphj(nary,kda);
jnpd=sphjp(nary,kda);
% legendre calculates the associated legendre polynomials.
% I'm only interested in the m=0 one
for term=nary
    lpn=legendre(term,cos(theta));
    pn(term+1,:)=lpn(1,:);
end
for term=1:size(theta,2)
    num=rhof*jnp.*jnd-kf*jnpd.*jn;
    denom=kf*jnpd.*hn-rhof*hnp.*jnd;
    intensity(term)=abs(-i/k*sum(nterm.*num./denom.*pn(:,term)'))^2;
end
end

```

Figures 11 through 14 are the squares of the scattering amplitudes for penetrable spheres.

5.3.2 Near Field

The same FORTRAN program used to compute near field intensities for the hard and soft spheres was also used with a penetrable sphere. Figure 15 shows the results of running this program for a sphere is radius 2.5 wavelengths with $k_d=1.2$ and $\rho_d=1.2$. An interesting feature is the focussing of the incident beam with a maximum intensity about 4 wavelengths beyond the center of the sphere.

5.4 Balloon

References

- [1] S. Turley, Scattering from a Circle, Internal Hughes Research Laboratory Notes, Aug. 1992.
- [2] S. Turley, 2d Scalar Surface Scattering, Internal Hughes Research Laboratory Notes, May 1992.
- [3] R. Kress, Acoustic Scattering, in *Scattering - Scattering and Inverse Scattering in Pure and Applied Science*, edited by E. Pike and P. Sabatier, chapter 2, Academic Press, London, 2001.
- [4] R. Kress, Acoustic Scattering, November 2006, a copy of sections 2.1 and 2.2 can be found at <http://diogenes.iwt.uni-bremen.de/vt/laser/papers/Kress-acoustic-scatter-ed-book.ps>.
- [5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Applied Mathematics Series, National Bureau of Standards, Cambridge, 1972.

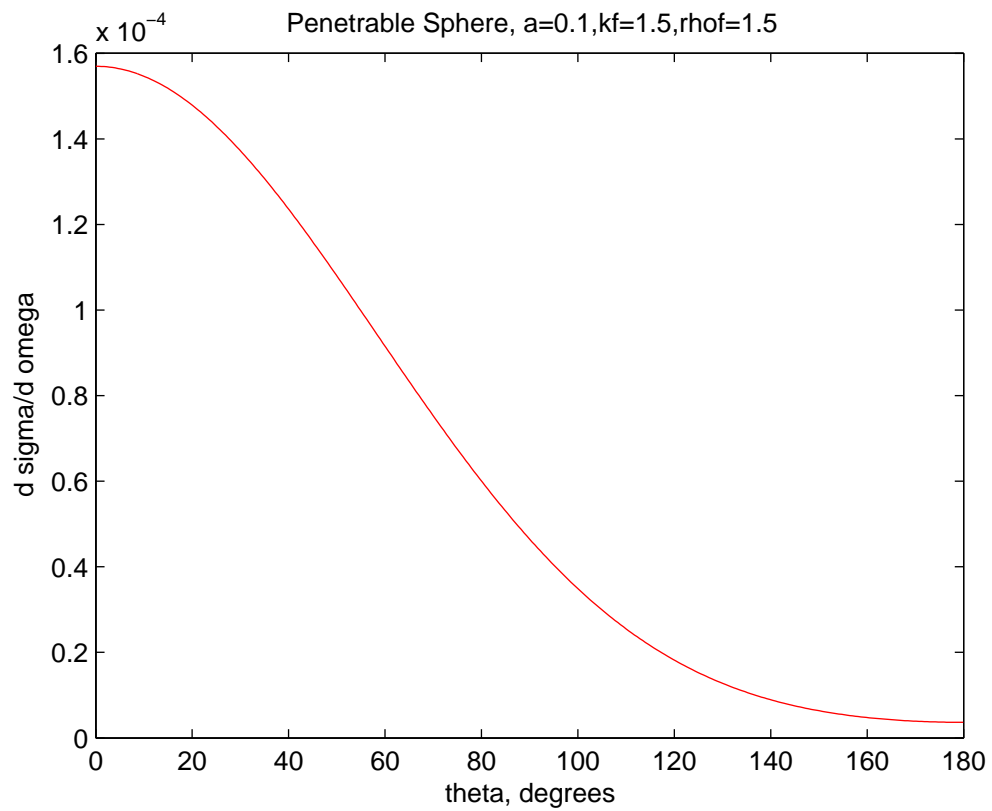


Figure 11: Differential cross section for scattering from a penetrable sphere of radius 0.1 wavelengths with $k_d/k = 1.5$ and $\rho_d/\rho = 1.5$.

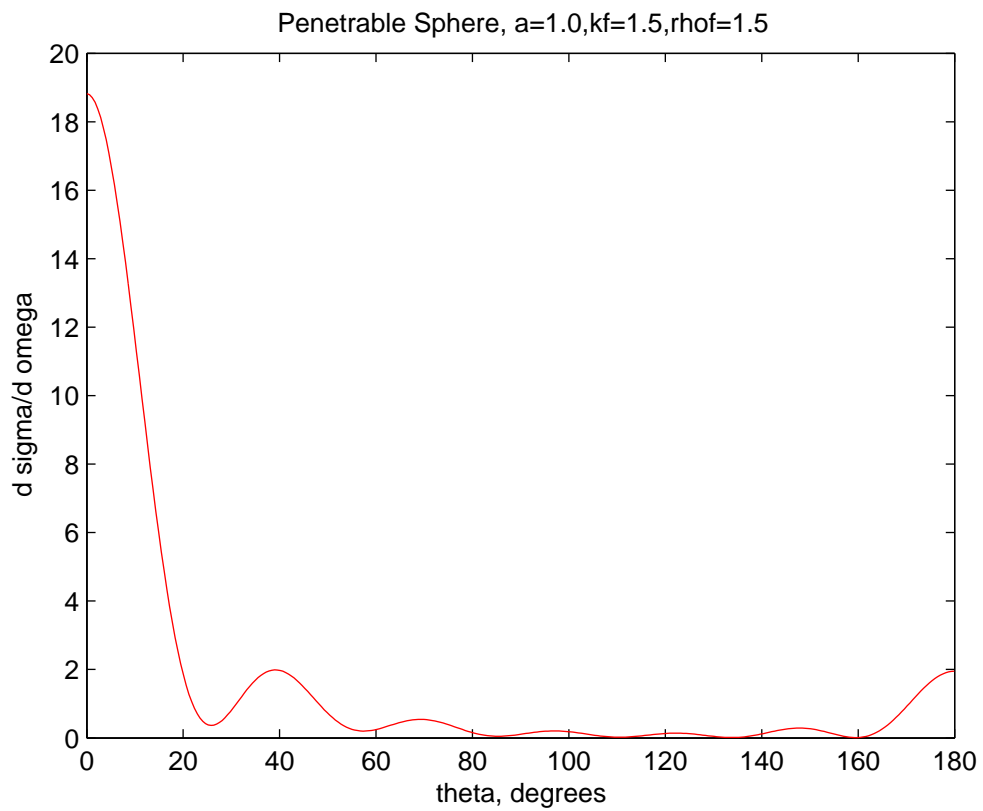


Figure 12: Differential cross section for scattering from a penetrable sphere of radius 1 wavelength with $k_d/k = 1.5$ and $\rho_d/\rho = 1.5$.

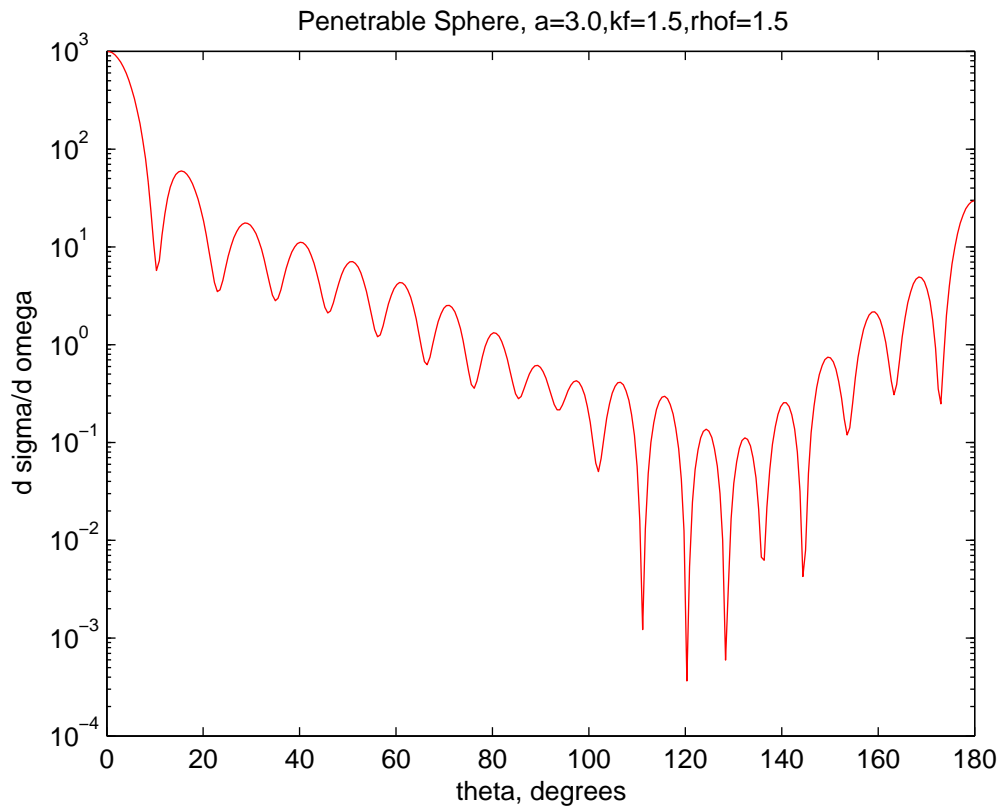


Figure 13: Differential cross section for scattering from a penetrable sphere of radius 3 wavelengths with $k_d/k = 1.5$ and $\rho_d/\rho = 1.5$.

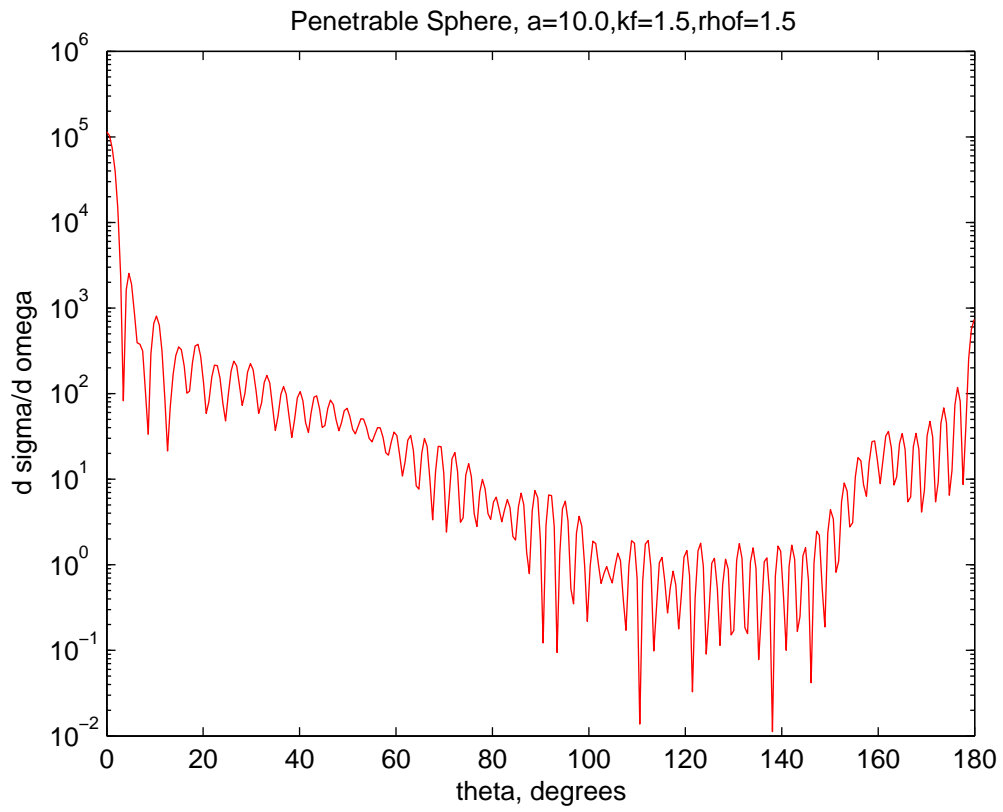


Figure 14: Differential cross section for scattering from a penetrable sphere of radius 10 wavelengths with $k_d/k = 1.5$ and $\rho_d/\rho = 1.5$.

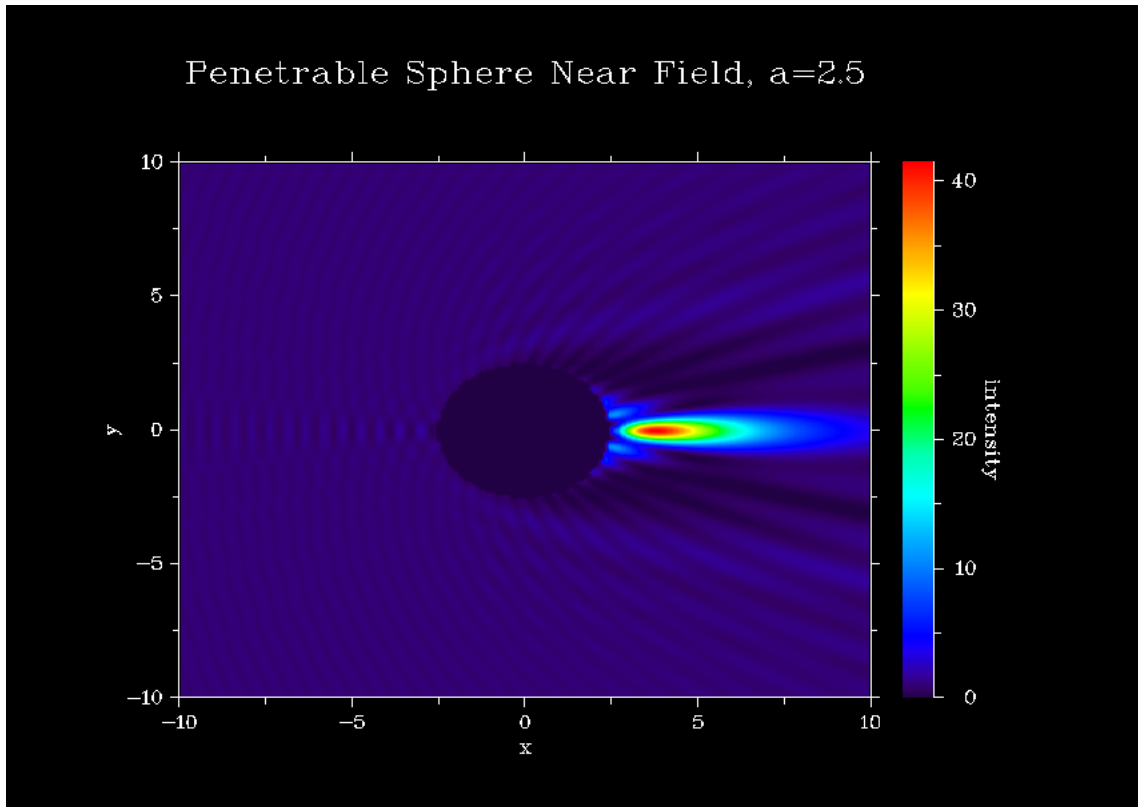


Figure 15: Total near field for scattering from a penetrable sphere of radius 2.5 wavelengths and with $k_d = 1.2$ and $\rho_d = 1.2$.